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# THE GROUPS OF ORDER $p^3q^{\beta^*}$

BY

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## § 1. *Introduction.*

The researches of FROBENIUS<sup>†</sup> and BURNSIDE,<sup>‡</sup> familiar to all students of the theory of groups, have established the non-existence of simple groups of orders  $pq^{\beta}$  and  $p^2q^{\beta}$ ,  $p$  and  $q$  being different primes, and the consequent solvability of all groups of these orders. In the following paper I show that all groups of order  $p^3q^{\beta}$  are compound, and therefore also solvable. For convenience of reference I place at the beginning of the discussion the two following theorems of which repeated use is made in the subsequent reasoning :

I. *A simple group  $\mathfrak{G}$  of order  $g = p^{\alpha}q^{\beta}$ ,  $p$  and  $q$  being different primes and  $p > q$ , cannot contain any subgroup  $\mathfrak{H}$  of order  $h$  whose index  $g/h$  in  $\mathfrak{G}$  is  $< p^2$ .*

In view of the results of FROBENIUS and BURNSIDE cited above, we assume that  $\alpha > 2$ .

The case  $g/h < p$  is trivial. Here every group  $\mathfrak{A}$  of order  $p^{\alpha}$  in  $\mathfrak{G}$  transforms every conjugate of  $\mathfrak{H}$  into itself and  $\mathfrak{G}$  is certainly compound. Again, for  $g/h = p$ ,  $h = p^{\alpha-1}q^{\beta}$ , every subgroup of order  $p^{\alpha-1}$  in  $\mathfrak{H}$  transforms  $\mathfrak{H}$  and therefore every conjugate of  $\mathfrak{H}$  into itself, and  $\mathfrak{G}$  is again certainly compound.

Suppose, now, that  $\mathfrak{R}$  is the largest subgroup of  $\mathfrak{G}$  that contains  $\mathfrak{H}$  and is less than  $\mathfrak{G}$ . Since  $\mathfrak{G}$  is simple,  $\mathfrak{R}$  is invariant under only those elements of  $\mathfrak{G}$  which are contained in  $\mathfrak{R}$ . The index  $g/k$  of  $\mathfrak{R}$  in  $\mathfrak{G}$  is  $\equiv g/h$  and  $> p$ ;  $\mathfrak{R}$  has  $g/k$  distinct conjugates in  $\mathfrak{G}$ .

1) Let  $g/k = pq^i$  ( $0 < q^i < p$ ),  $k = p^{\alpha-1}q^{\beta-i}$ . Having less than  $p^2$  conjugates in  $\mathfrak{G}$ ,  $\mathfrak{R}$  contains a subgroup of order  $p^{\alpha-1}$  from every group  $\mathfrak{A}$  of order  $p^{\alpha}$  in  $\mathfrak{G}$ . Every subgroup of order  $p^{\alpha-1}$  of  $\mathfrak{R}$  is permutable with every subgroup  $\mathfrak{L}$  of order  $g^{\beta-i}$  of  $\mathfrak{R}$ .  $\mathfrak{L}$  is invariant under a subgroup of  $\mathfrak{G}$  of order  $g^{\beta-i+1}$ , and this transforms  $\mathfrak{R}$  into at least one conjugate  $\mathfrak{R}'$  of  $\mathfrak{R}$ , different from  $\mathfrak{R}$ .

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† FROBENIUS: *Berliner Sitzungsberichte*, 1895, p. 185; cf. *Acta Mathematica*, vol. 26 (1902), p. 198.

‡ BURNSIDE: *Theory of Groups*, p. 348. Cf. JORDAN, *Liouville's Journal*, ser. 5, vol. 4 (1898), pp. 21-26.

$\mathfrak{K}$  and  $\mathfrak{K}'$ , having  $\mathfrak{L}$  in common, have no group of order  $p^{\alpha-1}$  in common. Then  $\mathfrak{L}$  is permutable with two different groups of order  $p^{\alpha-1}$  contained in the same group  $\mathfrak{A}$  of order  $p^\alpha$ , and is therefore permutable with  $\mathfrak{A}$ .  $\mathfrak{L}$  and  $\mathfrak{A}$  generate a group of order  $p^\alpha q^{\beta-i}$  contained in  $\mathfrak{G}$  and having  $\langle p \rangle$  conjugates in  $\mathfrak{G}$ .

2) Let  $g/k = q^j(p < q^j < p^2)$ ,  $k = p^\alpha q^{\beta-j}$ . As in 1), every conjugate of  $\mathfrak{K}$  contains a subgroup of order  $p^{\alpha-1}$  from every group  $\mathfrak{A}$  of order  $p^\alpha$  in  $\mathfrak{G}$ , and every subgroup  $\mathfrak{L}$  of order  $q^{\beta-j}$  of  $\mathfrak{K}$  occurs also in a conjugate  $\mathfrak{K}'$  of  $\mathfrak{K}$  different from  $\mathfrak{K}$ .  $\mathfrak{K}$  and  $\mathfrak{K}'$  have in common the group  $\mathfrak{L}$  and a group of order  $p^{\alpha-1}$  from every group  $\mathfrak{A}$  of  $\mathfrak{K}$  or  $\mathfrak{K}'$ ; their greatest common subgroup  $\mathfrak{D}$  is of order  $p^{\alpha-1}q^{\beta-j}$ . All the conjugates of  $\mathfrak{D}$  in  $\mathfrak{K}$  or  $\mathfrak{K}'$  are obtained by transforming  $\mathfrak{D}$  by any group  $\mathfrak{A}$  of  $\mathfrak{K}$  or  $\mathfrak{K}'$ . Hence all the subgroups of order  $p^{\alpha-1}$  in  $\mathfrak{D}$  are common to all the conjugates of  $\mathfrak{D}$  in  $\mathfrak{K}$  and  $\mathfrak{K}'$ . The subgroups of order  $p^{\alpha-1}$  of  $\mathfrak{D}$  generate a group invariant in  $\mathfrak{K}$  and in  $\mathfrak{K}'$ , and therefore in a group  $\mathfrak{M}$  contained in  $\mathfrak{G}$  and containing  $\mathfrak{K}$  and  $\mathfrak{K}'$ . Then  $\mathfrak{M} = \mathfrak{G}$ , and  $\mathfrak{G}$  is compound.

II. If a simple group  $\mathfrak{G}$  of order  $g = p^\alpha q^\beta$ ,  $p$  and  $q$  being, as in I, different primes and  $p > q$ , contains a subgroup  $\mathfrak{H}$  of order  $p^i q^j$  where  $1 < q^{\beta-j} < p$ , then  $\mathfrak{H}$  is contained in a subgroup  $\mathfrak{K}$  of  $\mathfrak{G}$  of order  $p^{i+x} q^\beta$  ( $x \geq 0, i+x < \alpha$ ).

Suppose that the largest group containing  $\mathfrak{H}$  and contained in  $\mathfrak{G}$  and  $\mathfrak{H}$  is  $\mathfrak{K}$  of order  $k = p^{i+x} q^{j+y}$  ( $j+y < \beta$ );  $\mathfrak{K}$  is invariant under only those elements of  $\mathfrak{G}$  that are contained in  $\mathfrak{K}$ ; having  $\langle p^{\alpha-i-x+1} \rangle$  conjugates in  $\mathfrak{G}$ ,  $\mathfrak{K}$  contains a subgroup of order  $p^{i+x}$  from every group  $\mathfrak{A}$  of order  $p^\alpha$  in  $\mathfrak{G}$ . If  $i+x = \alpha$ ,  $\mathfrak{G}$  is compound, by I. If  $i+x < \alpha$  and  $j+y < \beta$ , a subgroup  $\mathfrak{L}$  of order  $q^{j+y}$  of  $\mathfrak{K}$  occurs in a conjugate  $\mathfrak{K}'$  of  $\mathfrak{K}$  different from  $\mathfrak{K}$ .  $\mathfrak{L}$  is permutable with two groups of order  $p^{i+x}$  from the same group  $\mathfrak{A}$ ; these with  $\mathfrak{L}$  generate a group  $\mathfrak{M}$  of order  $p^{i+x+z} q^{j+y}$  ( $z > 0$ ) containing  $\mathfrak{K}$  and contained in  $\mathfrak{G}$  and  $\mathfrak{H}$ ; but this is contrary to assumption.

A simple application of Theorem II is afforded by the groups of order  $p^2 q^\beta$  ( $p > q > 2$ ). A simple group of this order must contain  $p^2$  subgroups  $\mathfrak{B}$  of order  $q^\beta$ ,  $p$  of which have a common subgroup  $\mathfrak{D}$  of order  $q^r$  ( $r > 0$ ) invariant in a group  $\mathfrak{D}'$  of order  $p q^{r+s}$  ( $s > 0$ );  $p^2 - 1$  is divisible by  $q^{\beta-r}$ , and  $p - 1$  by  $q^s$ , hence  $p - 1$  is divisible by  $q^{\beta-r}$ ;  $\mathfrak{D}'$  is contained in a subgroup of order  $p q^\beta$  of  $\mathfrak{G}$ . But then  $\mathfrak{G}$  is compound. The theorem controls also the case  $q = 2$ , except in the single event that  $s = 1$  and  $p + 1 = 2^{\beta-r-1}$ .

## § 2. Preliminary treatment of the groups of order $p^3q^8$ .

A simple group of order  $p^\alpha q^\beta$  ( $p \leq q$ ) can occur only if  $\alpha > 2\mu$ ,  $\mu$  being the lowest index for which  $p^\mu \equiv 1 \pmod{q}$ .\* For  $\alpha = 3$ , we can only take  $\mu = 1$ . A group  $\mathfrak{G}$  of order  $p^3 q^\beta$  can be simple only if  $p \equiv 1 \pmod{q}$ ; also  $\beta > 2$ , where  $q^\nu \equiv 1 \pmod{p}$ , therefore  $q^\beta > p^2$ .

\* BURNSIDE, *Theory of Groups*, p. 345. Cf. FROBENIUS, *Acta Mathematica*, vol. 26 (1902), p. 194.

A simple group  $\mathfrak{G}$  of order  $p^3q^\beta$  must contain either  $p^2$  or  $p^3$  subgroups  $\mathfrak{B}$  of order  $q^\beta$ . Since  $q^\beta > p^2$ , the elements of these subgroups  $\mathfrak{B}$  cannot be wholly distinct in either case.

If  $\mathfrak{G}$  contains only  $p^2$  subgroups  $\mathfrak{B}$ , and if two of these are so chosen that the order  $q^r$  of their greatest common divisor  $\mathfrak{D}$  is a maximum, then  $\mathfrak{D}$  is invariant in a subgroup  $\mathfrak{D}'$  of  $\mathfrak{G}$  whose order is  $p^xq^{r+s}$  ( $x = 1, 2$ ;  $s > 0$ ) and which contains exactly  $p$  groups of order  $q^{r+s}$ . Here  $p^2 - 1$  is divisible by  $q^{\beta-r}$ , therefore  $p + 1 \equiv q^{\beta-r-1}$ . (If  $q \neq 2$ ,  $p - 1$  is divisible by  $q^{\beta-r}$ , and  $\mathfrak{D}'$  has less than  $p^2$  conjugates in  $\mathfrak{G}$  unless  $x = 1$ . For odd  $q$ , the discussion can be greatly simplified, as in the case of order  $p^2q^\beta$ ). Each of the  $p^{3-x}q^{\beta-r-s}$  conjugate groups  $\mathfrak{D}$  occurs in exactly  $p$  of the groups  $\mathfrak{B}$ . If each of the  $p^2$  groups  $\mathfrak{B}$  contains  $k$  of the groups  $\mathfrak{D}$ , the total number of the groups  $\mathfrak{D}$  is  $p^2k/p = p^{3-x}q^{\beta-r-s}$ ; hence  $k = p^{2-x}q^{\beta-r-s}$ . If now  $x = 1$ , each group  $\mathfrak{B}$  contains  $pq^{\beta-r-s}$  groups  $\mathfrak{D}$ ; each of the latter is contained in  $p - 1$  other groups  $\mathfrak{B}$  and no two of them occur together in any second group  $\mathfrak{B}$ . But there are only  $p^2$  of the groups  $\mathfrak{B}$  and  $p^2 < pq^{\beta-r-s}(p - 1) + 1$ , unless  $q^{\beta-r-s} = 1$ ,  $r + s = \beta$ . Then  $\mathfrak{D}'$  is of order  $pq^\beta$ , has exactly  $p^2$  different conjugates in  $\mathfrak{G}$ , and therefore contains an element  $P$  of order  $p$  from every subgroup  $\mathfrak{A}$  of order  $p^3$  in  $\mathfrak{G}$ .

Let  $\mathfrak{A}$  be any subgroup of order  $p^3$  in  $\mathfrak{G}$ , and let  $\mathfrak{B}$  occur in  $\mathfrak{D}'$ ; having only  $p^2$  conjugates in  $\mathfrak{G}$ ,  $\mathfrak{B}$  is invariant under an element  $P$  of order  $p$  in  $\mathfrak{A}$ ;  $\mathfrak{B}$  is also permutable with a subgroup  $\mathfrak{A}_1$  of order  $p$  in  $\mathfrak{A}$  not containing  $P$ .  $P$  transforms  $\mathfrak{D}'$  into a conjugate of  $\mathfrak{D}'$  different from  $\mathfrak{D}'$  and containing the group  $P^{-1}\mathfrak{A}_1P$ , which is different from  $\mathfrak{A}_1$  but is contained with  $\mathfrak{A}_1$  in a subgroup  $\mathfrak{A}_2$  of order  $p^2$  of  $\mathfrak{A}$ .  $\mathfrak{B}$  is permutable with both  $\mathfrak{A}_1$  and  $P^{-1}\mathfrak{A}_1P$  and therefore with  $\mathfrak{A}_2$ ;  $\mathfrak{A}_2$  and  $\mathfrak{B}$  generate a group of order  $p^2q^\beta$  contained in  $\mathfrak{G}$  and having only  $p$  conjugates in  $\mathfrak{G}$ .

Again, if  $x = 2$  each group  $\mathfrak{B}$  contains  $q^{\beta-r-s}$  of the groups  $\mathfrak{D}$ . The  $p$  groups  $\mathfrak{B}$  which have subgroups of order  $q^{r+s}$  in a same group  $\mathfrak{D}'$  contain  $p(q^{\beta-r-s} - 1) + 1$  of the groups  $\mathfrak{D}$ , and these are transformed among themselves by every element of order  $p$  in  $\mathfrak{D}'$ . All the elements of order  $p$  in  $\mathfrak{D}'$  are therefore permutable with each of the remaining  $p - 1$  groups  $\mathfrak{D}$ ; they generate a group which is invariant in  $p$  groups  $\mathfrak{D}'$  and therefore in a group  $\mathfrak{M}$  of order  $p^{2+y}q^{r+s+t}$  ( $y = 0, 1$ ) contained in  $\mathfrak{G}$ . If  $y = 1$ ,  $\mathfrak{M}$  has at most  $p + 1$  conjugates in  $\mathfrak{G}$ . And if  $y = 0$ ,  $t > 0$  and  $\mathfrak{M}$  has at most  $p(p + 1)/q < p^2$  conjugates in  $\mathfrak{G}$ .

$\mathfrak{G}$  must therefore contain  $p^3$  sub-groups  $\mathfrak{B}$ . The maximum greatest common divisor  $\mathfrak{D}$ , of order  $q^r$ , of two of these is again invariant in a subgroup  $\mathfrak{D}'$  of order  $p^xq^{r+s}$  ( $x = 1, 2$ ) of  $\mathfrak{G}$ . Here  $p^3 - 1$  is divisible by  $q^{\beta-r}$ , and  $p - 1$ , being divisible by  $q$ , is divisible by  $q^{\beta-r-1}$  (in fact by  $q^{\beta-r}$  if  $q \neq 3$ ). Then, by the reasoning employed in the proof of Theorem II, if  $r + s < \beta$ ,  $\mathfrak{D}'$  is contained in a sub-group of order  $p^2q^\beta$  of  $\mathfrak{G}$ . Hence  $r + s = \beta$  and  $\mathfrak{D}'$  is of order

$pq^3$ . Each group  $\mathfrak{D}$  is common to  $p$  groups  $\mathfrak{B}$ . If each group  $\mathfrak{B}$  contains  $k$  groups  $\mathfrak{D}$ , we have  $p^3k/p = p^2$ , hence  $k = 1$ ; each group  $\mathfrak{D}'$  contains precisely one group  $\mathfrak{D}$ . Each group  $\mathfrak{B}$  occurs in only one group  $\mathfrak{D}'$ .

### § 3. Final Investigation of the Groups of Order $p^3q^3$ .

Each of the  $p^3$  groups  $\mathfrak{B}$  of  $\mathfrak{G}$  transforms among themselves the  $p^3 - p$  conjugates of  $\mathfrak{B}$  not contained in the group  $\mathfrak{D}'$  in which  $\mathfrak{B}$  occurs. Let  $\mathfrak{D}'$ , and  $\mathfrak{B}$  in  $\mathfrak{D}'$ , be so chosen that a subgroup  $\Delta$  common to  $\mathfrak{B}$  and a conjugate of  $\mathfrak{B}$  not contained in  $\mathfrak{D}'$  is of the largest possible order, and let this order be  $q^\rho$ . Then  $q^{\beta-\rho}$  divides  $p^3 - p$ , and therefore divides  $p^2 - 1$ ;  $\rho > 0$ , and in general  $p > q^{\beta-\rho-1}$ , the only exception occurring when  $q = 2$  and  $p + 1 = 2^{\beta-\rho-1}$ . In this exceptional case  $\beta - r = 1$ ,  $\mathfrak{D}$  is of order  $2^{\beta-1}$ .

The group  $\Delta$  is common to two groups  $\mathfrak{B}$  from different groups  $\mathfrak{D}'$ .  $\Delta$  is invariant under subgroups  $\mathfrak{N}_1, \mathfrak{N}_2$  of order  $q^{\rho+\sigma_1}, q^{\rho+\sigma_2}$  ( $\sigma_1, \sigma_2 > 0$ ) of these two groups  $\mathfrak{B}$ .  $\mathfrak{N}_1$  and  $\mathfrak{N}_2$  cannot be contained in any subgroup of  $\mathfrak{G}$  of order  $q^{\rho+\tau}$  ( $\tau > 0$ ), for this subgroup would be common to two groups  $\mathfrak{D}'$  and therefore to two groups  $\mathfrak{B}$  contained one in each of these two groups  $\mathfrak{D}'$ .  $\Delta$  is invariant in a subgroup  $\Delta'$  of order  $p^xq^{\rho+\sigma}$  ( $x, \sigma > 0$ ) of  $\mathfrak{G}$ . Any subgroup of order  $p^x$  of  $\Delta'$  transforms  $\mathfrak{D}'$  containing  $\Delta$  into precisely  $p$  groups  $\mathfrak{D}'$  each containing  $\Delta$ , for  $\Delta$  cannot occur in all the  $p^2$  groups  $\mathfrak{D}'$ .  $\Delta'$  has one or more subgroups of order  $q^{\rho+\sigma}$  common with each of these  $p$  groups  $\mathfrak{D}'$ , and no subgroup of order  $q^{\rho+\tau}$  ( $\tau > 0$ ) common with any other group  $\mathfrak{D}'$ . Hence  $\Delta$  occurs in precisely  $p$  groups  $\mathfrak{D}'$ .

1) If  $p > q^{\beta-\rho-1}$ , or if  $\sigma > 1$ , then by Theorem II,  $\Delta'$  is contained in a subgroup  $\mathfrak{M}$  of order  $pq^3$  of  $\mathfrak{G}$  (the order  $p^2q^3$  being inadmissible).  $\mathfrak{M}$  contains  $p$  groups  $\mathfrak{B}$  having  $\Delta$  as their common subgroup;  $\Delta$  is invariant in  $\mathfrak{M}$ ,  $\mathfrak{M} = \Delta'$ ,  $\rho + \sigma = \beta$ ; and  $\Delta$  has  $p^2$  conjugates in  $\mathfrak{G}$ . If each group  $\mathfrak{D}'$  contains  $k$  groups  $\Delta$ , then since each group  $\Delta$  occurs in  $p$  groups  $\mathfrak{D}'$  we have  $p^2k/p = p^2$ , hence  $k = p$ .

If now two groups  $\mathfrak{D}'$  have more than one group  $\Delta$  in common, their greatest common divisor  $\mathfrak{C}$  is of order  $pq^\rho$  and contains  $p$  groups  $\Delta$ ; these are all the groups  $\Delta$  contained in the two groups  $\mathfrak{D}'$ ;  $\mathfrak{C}$  is the smallest group that contains them;  $\mathfrak{C}$  is invariant in both groups  $\mathfrak{D}'$  and has  $< p^2$  conjugates in  $\mathfrak{G}$ .

If no two groups  $\mathfrak{D}'$  have more than one group  $\Delta$  in common, the  $p$  groups  $\Delta$  contained in any group  $\mathfrak{D}'$  are distributed among  $p(p-1)+1$  groups  $\mathfrak{D}'$ , and none of them occur in  $p-1$  groups  $\mathfrak{D}'$ . All the elements of order  $p$  of a group  $\mathfrak{D}'$  are therefore permutable with each of  $p-1$  other groups  $\mathfrak{D}'$ ; these elements of order  $p$  are common to  $p$  groups  $\mathfrak{D}'$  and are all the elements of order  $p$  of any of these  $p$  groups  $\mathfrak{D}'$ ; they generate a group invariant under the  $p$  groups  $\mathfrak{D}'$  and having  $< p^2$  conjugates in  $\mathfrak{G}$ .

2) It remains to consider the special case  $q = 2$ ,  $p + 1 = 2^{\beta-\rho-1}$ ,  $\sigma = 1$ ,  $r = \beta - 1$ . Let  $\mathfrak{D}'_1, \mathfrak{D}'_2$  be two groups  $\mathfrak{D}'$  having a group  $\Delta$  in common.  $\Delta$

cannot be the greatest common divisor of  $\mathfrak{D}'_1$  and  $\mathfrak{D}'_2$ , since either of the latter would then transform the other into  $p2^{3-p} > p^2$  conjugates. The greatest common divisor  $\mathfrak{C}$  of  $\mathfrak{D}'_1$ ,  $\mathfrak{D}'_2$  is therefore of order  $p2^p$ .

Suppose first that  $\mathfrak{C}$  contains only one group of order  $2^p$ , that is, that  $\Delta$  is invariant in  $\mathfrak{C}$ ; then  $\Delta'$  contains  $\mathfrak{C}$ . If  $\Delta'$  were of order  $p2^{p+1}$ ,  $\mathfrak{C}$  would be invariant in  $\Delta'$  and would contain every element of order  $p$  of  $\Delta'$ ; whereas  $\Delta'$  must contain an element of order  $p$  or  $p^2$  which transforms  $\mathfrak{D}'_1$  into  $\mathfrak{D}'_2$ . Hence  $\Delta'$  is of order  $p^22^{p+1}$ , and  $\Delta$  has  $p2^{3-p-1} = p(p+1)$  conjugates in  $\mathfrak{G}$ . If each group  $\mathfrak{D}'$  contains  $k$  groups  $\Delta$ , we have  $p^2k/p = p(p+1)$ ,  $k = p+1$ . Since  $\Delta$  is invariant under an element  $p$  of  $\Delta'$ ,  $\Delta$  is contained in the group  $\mathfrak{D}$  occurring in  $\mathfrak{D}'$ ; the  $p+1$  groups  $\Delta$  occurring in  $\mathfrak{D}'$  are conjugate in  $\mathfrak{D}'$  and are therefore all contained in  $\mathfrak{D}$ . No two groups  $\Delta$  occurring in the same group  $\mathfrak{D}'$  can occur together in any second group  $\mathfrak{D}'$ . The  $p$  groups  $\mathfrak{D}'$  which have  $\Delta$  in common, contain  $p^2+1$  groups  $\Delta$ , and do not contain any one of the remaining  $p-1$  groups  $\Delta$ .  $\Delta'$  transforms among themselves the  $p^2+1$  groups  $\Delta$  occurring with  $\Delta$  in groups  $\mathfrak{D}'$ , and transforms among themselves the remaining  $p-1$  groups  $\Delta$ . All the elements of order  $p$  or  $p^2$  in  $\Delta'$  are permutable with these  $p-1$  groups  $\Delta$ ; they all occur in  $p$  groups  $\Delta'$  and are all the elements of order  $p$  or  $p^2$  of each of these groups  $\Delta'$ ; they generate a group invariant under  $p$  groups  $\Delta'$  and having  $< p^2$  conjugates in  $\mathfrak{G}$ .

The group  $\mathfrak{C}$  common to any two groups  $\mathfrak{D}'_1$  and  $\mathfrak{D}'_2$  must therefore contain  $p$  groups  $\Delta$ . No group  $\Delta$  is contained in a group  $\mathfrak{D}$ , for then  $\Delta$  would be invariant in  $\mathfrak{C}$ . Since  $\mathfrak{D}$  is of order  $2^{3-p}$ ,  $\Delta$  has a subgroup  $\mathfrak{S}$  of order  $2^{p-1}$  common with  $\mathfrak{D}$ .  $\mathfrak{S}$  is invariant in  $\mathfrak{C}$ , since  $\mathfrak{C}$  cannot have two groups of order  $2^{p-1}$  common with  $\mathfrak{D}$ .  $\mathfrak{S}$  is also invariant under subgroups of order  $2^{p+\tau_1}$ ,  $2^{p+\tau_2}$  ( $\tau_1, \tau_2 \geq 0$ ) of  $\mathfrak{D}_1$  and  $\mathfrak{D}_2$ , and therefore under subgroups  $\mathfrak{L}_1$ ,  $\mathfrak{L}_2$  of order  $p2^{p+\tau_1}$ ,  $p2^{p+\tau_2}$  ( $\tau_1, \tau_2 > 0$ ) of  $\mathfrak{D}'_1$  and  $\mathfrak{D}'_2$  respectively.  $\mathfrak{L}_1$  and  $\mathfrak{L}_2$  cannot both be contained in a group of order  $p2^{p+\tau}$  containing  $\mathfrak{C}$ . The largest group  $\mathfrak{S}'$  contained in  $\mathfrak{G}$  and containing  $\mathfrak{S}$  as invariant subgroup is therefore of order  $p^22^{p+\tau}$ , where we must take  $\tau = 1$ , by virtue of Theorem II.  $\mathfrak{S}'$  transforms  $\mathfrak{D}'$  containing  $\mathfrak{S}$  into  $p$  groups  $\mathfrak{D}'$  containing  $\mathfrak{S}$  and has  $p2^{p+1}$  elements in common with each of these  $p$  groups  $\mathfrak{D}'$ .  $\mathfrak{S}$  does not occur in any other group  $\mathfrak{D}'_i$ . For any group of order  $p^2$  in  $\mathfrak{S}'$  could transform  $\mathfrak{D}'_i$  into only  $p$  groups  $\mathfrak{D}'$ ;  $\mathfrak{S}'$  would have an element  $P$  of order  $p$  common with  $\mathfrak{D}'_i$ ;  $\mathfrak{S}$ , being invariant under  $P$ , would be contained in  $\mathfrak{D}_i$  and would be invariant under a group of order  $2^p$  of  $\mathfrak{D}_i$ ; this group of order  $2^p$  would occur in  $\mathfrak{S}'$  and therefore in one of the  $p$  groups  $\mathfrak{D}'$  into which  $\mathfrak{S}'$  transforms  $\mathfrak{D}'_i$ ; and this would lead to the case already disposed of where  $\mathfrak{C}$  contains only one group  $\Delta$ .

The group  $\mathfrak{S}$  has  $p(p+1)$  conjugates in  $\mathfrak{G}$  and  $p+1$  conjugates in each group  $\mathfrak{D}'$ . The  $p(p+1)$  groups  $\mathfrak{S}'$  are all different. The  $p+1$  conjugates of  $\mathfrak{S}$  which occur in  $\mathfrak{D}'$  are conjugate in  $\mathfrak{D}'$  and are all contained in  $\mathfrak{D}$ . No

two groups  $\mathfrak{D}'$  can have two groups  $\mathfrak{S}$  in common, since this would again lead to the rejected case where  $\mathfrak{G}$  contains only one group  $\Delta$ . The  $p$  groups  $\mathfrak{D}'$  which have  $\mathfrak{S}$  in common contains  $p^2 + 1$  groups  $\mathfrak{S}$ .  $\mathfrak{S}'$  transforms these  $p^2 + 1$  groups  $\mathfrak{S}$  among themselves. All the elements of order  $p$  or  $p^2$  in  $\mathfrak{S}'$  are permutable with each of the remaining  $p - 1$  groups  $\mathfrak{S}$ ; they generate a group invariant in  $p$  groups  $\mathfrak{S}'$  and having  $< p^2$  conjugates in  $\mathfrak{G}$ .

COLUMBIA UNIVERSITY,  
*January, 1904.*

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